5 Mathematics

5.1 Learning outcomes

After studying this text the learner should / should be able to:

- Know, understand and be able to calculate future value from present value and vice versa.
- Understand how the annuity formula applies to bonds.
- Know, understand and be able to calculate the price (value) of a bond with a fixed coupon rate and a fixed term to maturity.
- Understand the concepts of cum interest and ex interest.
- Know, understand and be able to apply the standard price formula for bonds with 6 months or longer to maturity and for bonds with less than 6 months to maturity.
- Know, understand and be able to apply the following formulae: perpetual bonds, bonds with a variable rate, CPI bonds, zero coupon bonds and strips.

5.2 Introduction

The basic tenet of the time value of money concept is that a LCC received today is worth more than a LCC received at some stage in the future, i.e. money has a *future value* (FV) and a *present value* (PV). FV is present value plus interest, and PV is future value discounted by a relevant rate.

Another basic principle of the concept is that interest is compounded, i.e. interest that is earned is reinvested, and an essential assumption here is that interest earned is reinvested at the rate earned on the principal amount.

The PV / FV concept is the foundation of virtually all the mathematics of the financial markets. "Virtually" is used because it does not play a role in certain derivatives. The aim of this chapter is to elucidate the mathematics of the bond market. The following is covered:

- Present value / future value.
- Annuities.
- Plain vanilla bonds.
- Perpetual bonds.
- Bonds with a variable rate.
- CPI bonds.
- Zero coupon bonds.
- STRIPS.

5.3 Present value / future value

The FV of an investment is calculated in terms of the following formula:

$$FV = PV\,(1 + ir/cp)^{y.cp}$$

where

ir = interest rate pa expressed as a fraction of 1
y = number of years
cp = number of compounding periods pa (number of times interest is paid)

Example: PV = LCC1 000 000
ir = 0.15
y = 3
cp = 2 (i.e. six-monthly)

$$\begin{aligned} FV &= \text{LCC1 000 000}\,(1 + 0.15/2)^{3x2} \\ &= \text{LCC1 000 000}\,(1.075)^{6} \\ &= \text{LCC1 000 000}\,(1.54330153) \\ &= \text{LCC1 543 301.53.} \end{aligned}$$

This means that an investment made now of LCC1 000 000 at 15% pa, compounded six-monthly, for 3 years will be worth LCC1 543 301.53 then.

The PV formula is derived from the future value formula:

$$PV = FV / (1 + ir/cp)^{y.cp}$$

Example:

$FV = LCC1\ 000\ 000$
$ir = 0.12$
$y = 3$
$cp = 2$ (i.e. six-monthly)

$$\begin{aligned} PV &= LCC1\ 000\ 000 / (1 + 0.12/2)^{3.2} \\ &= LCC1\ 000\ 000 / (1.06)^{6} \\ &= LCC1\ 000\ 000 / 1.41851911 \\ &= LCC704\ 960.54. \end{aligned}$$

This answers the question: what amount must be invested now (PV) at 12% per annum compounded semi-annually to end up at LCC1million in 3 years' time?

The next step in elucidating bond market mathematics is the PV / FV concept inherent in the annuity calculation.

5.4 Annuities

The coupon payments of a fixed-rate bond are identical to the cash flows of an annuity; therefore an understanding of the principles of annuities is required.

An annuity is an investment now (PV) that comprises regular *payments* of the same amount over a stipulated period in the future. This is called a *fixed annuity*. The *life annuity* does of course exist (where the period of payments is not certain), but this we will leave to actuarial science.

The PV of a *stream of fixed payments* (PMT) in the future (various FVs of the same amount) is determined according to the following formula:

$$PV = [PMT / (1 + ir/cp)^{y.cp}] + [PMT / (1 + ir/cp)^{y.cp}] \ldots$$

It follows that the PV of a 3-year annuity with *annual* payments is calculated according to:

$$PV = [PMT / (1 + ir)^1] + [PMT / (1 + ir)^2] + [PMT / (1 + ir)^3].$$

Example 1:

PMT = LCC200 000
ir = 9% pa
y = 3
cp = 1 (remember cp = compounding periods pa)

$$PV = [LCC200\,000 / (1.09)^1] + [LCC200\,000 / (1.09)^2] + [LCC200\,000 / (1.09)^3]$$
$$= (LCC200\,000 / 1.09) + (LCC200\,000 / 1.1881) + (LCC200\,000 / 1.2950)$$
$$= LCC183\,486.24 + LCC168\,336.00 + LCC154\,440.15$$
$$= LCC506\,262.39.$$

The PV of a 2-year annuity with semi-annual payments is calculated according to:

$$PV = [PMT / (1 + ir/2)^{.5\times2}] + [PMT / (1 + ir/2)^{1\times2}] + [PMT / (1 + ir/2)^{1.5\times2}] + [PMT / (1 + ir/2)^{12\times2}].$$

Example 2: PMT = LCC100 000
ir = 9% pa
y = 2
cp = 2

$$
\begin{aligned}
PV &= (LCC100\,000 / (1.09/2)^{.5\times.2}) + (LCC100\,000 / (1.09/2)^{1\times2}) + \\
&\quad (LCC100\,000 / (1.09)^{1.5\times2}) + (LCC100\,000 / (1.09/2)^{2\times2}) \\
&= (LCC100\,000 / (1.045)^{1}) + (LCC100\,000 / (1.045)^{2}) + \\
&\quad (LCC100\,000 / (1.045)^{3}) + (LCC100\,000 / (1.045)^{4}) \\
&= (LCC100\,000 / 1.045) + (LCC100\,000 / 1.092025) + \\
&\quad (LCC100\,000 / 1.141166) + (LCC100\,000 / 1.192519) \\
&= LCC95\,693.78 + LCC91\,573.00 + LCC87\,629.67 + LCC83\,856.11 \\
&= LCC358\,752.56.
\end{aligned}
$$

It will be apparent that the PV is the value now of the fixed annuity payments in the future, i.e. the PV is derived from the future payments (FV) and the interest rate applicable for the period. This means that a person has to pay (in the case of the last example) LCC358 752.56 now in order to receive payments of LCC100 000 every six months for two years. It will also be clear that what was done above was to *discount* each payment by the spot rate of interest for the period and then to add them.

The principle applied here is applicable to plain vanilla long-term bonds. A plain vanilla bond is nothing else than a finite series of fixed payments made in the future and repayment of the face value of the bond on redemption date.

5.5 Plain vanilla bond

5.5.1 Introduction

The plain vanilla bond is the most common one – it pays a fixed rate of interest at regular intervals and the nominal / face value at maturity / redemption. The intervals may be quarterly, six-monthly or annually in arrears, but the six-monthly one is the most universal. These bonds may be issued at one of three prices:

- 100% (=1.0)[29].
- Lower than 100% (< 1.0).
- Higher than 100% (> 1.0).

If issued at 100%, i.e. at par value, the coupon is the "rate" earned by the holder. If issued at a discount to par value, the "rate" earned is higher than the coupon. Clearly then, if a bond is issued at a premium, the "rate" earned is lower that the coupon.

To illustrate the principle, let us use a ridiculous example: assume that a LCC1 million (nominal value) bond has 365 days to run and one coupon payable at the expiry of the bond of 12% pa. Clearly, if this bond is issued at a price of 100%, the investor will earn 12% pa. If, however, the bond is issued at a discount price of, say, 98.3%, the investor will pay LCC983 000 (LCC1 000 000 × 0.983). However, s/he will earn the coupon of 12%, i.e. LCC120 000 (LCC1 000 000 × 0.12). Thus, his/her return is actually 12.21% pa (LCC120 000 / LCC983 000 × 100).

If the bond is issued at a premium of 105.2%, the investor will pay LCC1 052 000 (LCC1 000 000 × 1.052), but will earn LCC120 000 at the end of the period. The return is then 11.41% (LCC120 000 / LCC1 052 000 × 100).

As noted, these are ridiculous examples because bonds are longer in term and coupons are usually paid six-monthly. This makes them more complicated in terms of their mathematics. It will also have been noted above that use of the word *rate* in connection with discount and premium was placed in inverted commas. The reason is that in the bond market one cannot talk of the "rate" earned. It is more complex: in this respect market participants talk of the *yield to maturity* or the *yield to redemption*. We will use the former.

3.5.2 Yield to maturity

The yield to maturity (ytm) is a measure of the *rate of return* on a bond that has a number of coupons paid over a number of years and a face value payable at maturity. It may also be described as the rate that buyers are prepared to pay now (present value LCC) for a stream of regular payments and a lump sum at the end of the period for which the bond is issued. It is an *average rate* earned per annum over the period.

Formally described, the ytm is the *discount rate that equates the future coupon payments and principal amount of a bond with the market price*. Another way of stating this is: the *price* is merely the *discounted value of the income stream (ie the coupon payments and redemption amount), discounted at the market yield (ytm).*

A basic example will illuminate:

Settlement date:	30/9/2005
Maturity date:	30/9/2008
Coupon rate:	9% pa
Face value:	LCC1 000 000
Interest date:	30/9
ytm	8% pa (payable annually in arrears).

The cash flows and their discounted values (using the ytm) are as shown in Table 1.

Date	Coupon Payment	Face value	Compounding periods (cp)	Present value $C / (1 + ytm)^{cp}$
30/9/2006	LCC90 000	-	1	LCC83 333.33
30/9/2007	LCC90 000	-	2	LCC77 160.49
30/9/2008	LCC90 000	-	3	LCC71 444.90
30/9/2008	-	LCC1 000 000	3	LCC793 832.24
Total	**LCC270 000**	**LCC1 000 000**		**LCC1 025 770.96**
C = coupon. cp = compounding periods.				

Table 1: Cash flows and discounted values

The value now of the bond is LCC1 025 770.96, and the price of the bond is 1.02577096 or 102.577096%. It will be apparent that the price is derived from the following formula for bonds:

$$\text{Price (PV)} = [cr / (1 + ytm)^1] + [cr / (1 + ytm)^2] + [cr / (1 + ytm)^3] + [1 / (1 + ytm)^3]$$

where:

cr = coupon rate pa (expressed as a fraction of 1)
ytm = yield to maturity (expressed as a fraction of 1).

Using the same numbers as above (coupon rate = 9% pa, ytm = 8% pa):

Price (PV) = (0.09 / 1.08) + (0.09 / 1.166400) + (0.09 / 1.259712) + (1 / 1.259712)
= 0.08333333 + 0.07716049 + 0.0714449 + 0.79383224
= 1.02577096.

It will be apparent that the coupon rate (0.09) for the periods and the face value (1) that takes place at maturity (all FVs) are discounted at the ytm to PV. Because the coupon rate is higher than the ytm, the price is *higher than 1* (a premium to par). Where the coupon rate is equal to the ytm (assume 9% pa) the price is *equal to 1* (par):

Price (PV) = (0.09 / 1.09) + (0.09 / 1.1881) + (0.09 / 1.295029) + (1 / 1.295029)
= 0.082569 + 0.075751 + 0.069497 + 0.772183
= 1.000000.

As noted, when the coupon rate is lower than the ytm (assume coupon rate = 9% pa, ytm = 11% pa), the price is *lower than 1* (i.e. at a discount to par):

Price (PV) = (0.09 / 1.11) + (0.09 / 1.232100) + (0.09 / 1.367631) + (1 / 1.367631)
= 0.081081 + 0.073046 + 0.065807 + 0.731191
= 0.951125.

The *inverse relationship* between ytm and price will also be clear. This is because the ytm is the *denominator* in the formula. Thus, if the ytm falls, the price of the bond rises. It follows that if the ytm increases the price falls. Another way of seeing this phenomenon is the logic of: as the ytm rises the future cash flows are worth less when discounted to present value, pulling down the price.

In reality bonds are slightly more complicated but the principle remains the same. The majority (by far) of bonds issued in the bond market have coupons that are payable six-monthly in arrears, and they are issued and traded for periods that are broken, i.e. issues and secondary market settlement dates are between interest payment dates.

In the case where interest payments are made six-monthly in arrears (ignoring settlement between interest payment dates), the coupon rate is halved and the compounding periods are doubled (assume a three-year bond):

$$\begin{aligned} \text{Price (PV)} \quad = \quad & [(cr/2) / (1 + ytm/2)^1] + [(cr/2) / (1 + ytm/2)^2] + \\ & [(cr/2) / (1 + ytm/2)^3] + [(cr/2) / (1 + ytm/2)^4] + \\ & [(cr/2) / (1 + ytm/2)^5] + [(cr/2) / (1 + ytm/2)^6) + [1 / (1 + ytm/2)^6]. \end{aligned}$$

The aforementioned bond formula is usually written as:

$$\text{Price} = \sum_{t=1}^{n} [cr / (1 + ytm)^t] + [1 / (1 + ytm)^n]$$

where

cr = coupon rate (cr / 2 if six-monthly)
ytm = yield to maturity (ytm / 2 if six-monthly)
n = number of periods (years × 2 if six-monthly).

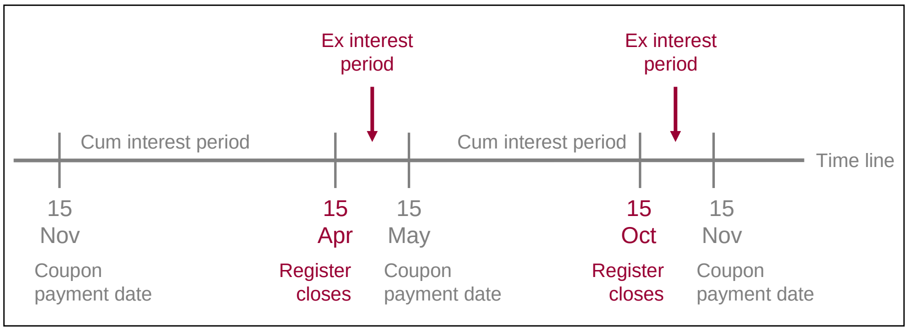

Figure 1: cum and ex interest

Another concept that needs grasping before proceeding to the more elaborate bond formula is that of *cum interest* and *ex interest*. This arises out of the fact that bond registers close for a period prior to the interest payment dates (we assume one month[30]). The date on which the register closes is known as the *last day to register*. Prior to this date, the bond is said to be *cum interest* and from the last day to register until the date of the interest payment it is said to be *ex interest* (see Figure 1).

If a bond is sold in the *cum interest* period the ownership change is recorded in the register and the new investor receives the full half coupon. S/he has to compensate the previous holder for the interest accrued to him/her during the relevant period. This interest factor is added to the clean price in order to arrive at the all-in price (also called *dirty price*). For example, if the coupon is 10% pa, and the bond was sold on 16 July, the price the new investor pays is increased by 0.016986 (62/365 × 0.10).[31]

If a deal is settled on say 16 October, the bond remains registered in the name of the "old" holder until the interest payment date of 15 November. The new holder must be compensated for the interest not received that accrues to him/her, i.e. interest from 16 October to 15 November. The applicable interest factor, 0.008493 (31/365 × 0.10), is deducted from the clean price in order to arrive at the all-in price (dirty price).

Thus, as can be seen in Figure 2 (assumption: no market rate change over the period), in ex interest periods the dirty price is below the clean price (= buyer pays lower price / gets higher rate – because seller gets the full coupon), whereas in cum interest periods the dirty price is above the clean price (= buyer pays higher price / earns lower rate – because s/her gets the full coupon).

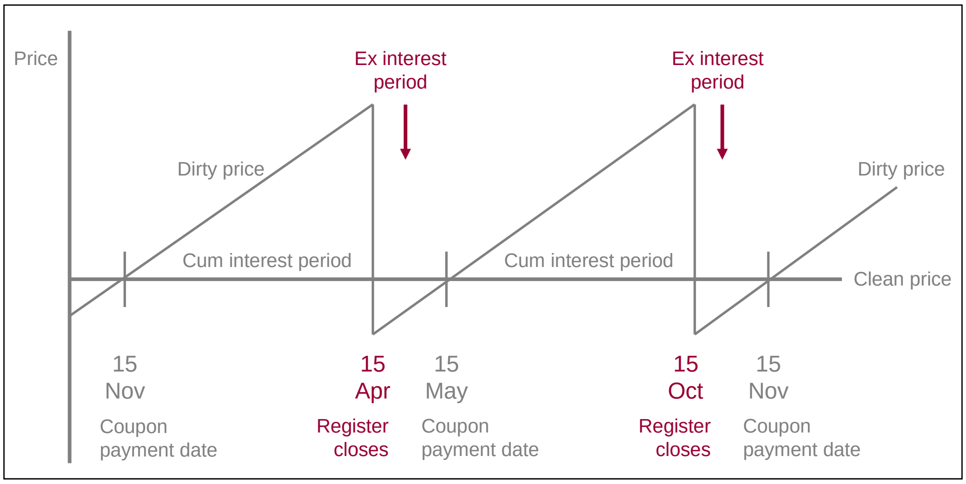

Figure 2: cum and ex interest

5.5.3 Standard pricing formula

The standard pricing formula for plain vanilla bonds (with more than 6 months to maturity) takes into account all the possibilities referred to above. It is:

$$\text{All-in price} = V_i^{d1/d2}\,[1/2\ c\ (a_n^{\ i} + e) + 100V_i^{\ n}]$$

where

d1 = number of days from settlement date to next interest date

d2 = number of days from last to next interest date or from settlement date to next interest date if settlement falls on an interest date

i = yield at which bond trades (percentage)

V_i = 1 / (1 + i / 200)
= present value of 1 payable in 6 months' time

c = coupon (percentage)

n = number of full six month periods from next interest date to redemption date

$a_n^{\ i}$ = $(1 - V_i^{\ n}) / (i / 200)$
= present value of an annuity of 1 per six months, payable in arrears

e = 1 if the stock is cum interest and 0 if ex interest

Accrued interest = [(d2 – d1) / 365] × c (if cum interest)
Accrued interest = – (d1 / 365 × c) (if ex interest)
Clean price = all-in price – accrued interest.

The following example may be useful:

Coupon (payable half-yearly)	= 12% pa
Coupon payment dates	= 15 March and 15 September
Redemption (maturity) date	= 15 September 2009
Yield to maturity	= 13.5% pa
Settlement date	= 20 July 2005
Nominal value	= LCC1 000 000.

In this example the all-in price is 99.4450610, and the consideration (amount paid) would be LCC994 450.61. It will be evident that this bond is *cum interest.* The following will be useful:

d1	= 20 July 2005 – 15 September 2005	= 57 days
d2	= 15 March 2005 – 15 September 2005	= 184 days
n	= 8	
d1/d2	= 0.30978261	
V_i	= 0.93676815	
V_i^n	= 0.59300306	
$V_i^{d1/d2}$	= 0.97996852	
$a_n^{\ i}$	= 6.02958430	
All-in price	$= V_i^{d1/d2} \times [1/2c \times (a_n^{\ i} + e) + (100 \text{ x } V_i^n)]$	
	$= 0.97996852 \times [6.0 \times (6.02958430 + 1) + 59.300306]$	
	$= 0.97996852 \times 101.4778118$	
	= 99.4450610.	

If, however, the bond was purchased on 20 August 2005, the all-in price would be 94.60061322 and the consideration LCC946 006.13. It will be evident that this bond is *ex interest*. The following will be useful:

d1	= 20 August 2005 – 15 September 2005 = 26 days	
d2	= 15 March 2005 – 15 September 2005	= 184 days
n	= 8	
d1/d2	= 0.14130435	
V_i	= 0.93676815	
V_i^n	= 0.59300306	
$V_i^{d1/d2}$	= 0.99081254	
$a_n^{\ i}$	= 6.02958430	
All-in price	$= V_i^{d1/d2} \times [1/2c \times (a_n^{\ i} + e) + (100 \times V_i^n)]$	
	$= 0.99081254 \times [6.0 \times (6.02956430 + 0) + 59.300306]$	
	$= 0.99081254 \times 95.4778118$	
	= 94.60061322.	

In a number of countries it is also a convention to calculate and present clients with the accrued interest and the so-called *clean price* (mainly for purposes of exchange control and to calculate the so-called *running yield*). As noted above, the *clean price* is determined by deducting the accrued interest from the all-in price.

In the above example the accrued interest price is equal to:

Accrued interest price = (184 – 57) / 365 × c
= 4.17534.

Clean price = All-in price – accrued interest price
= 99.44506 – 4.17534
= 95.26972.

In the case of an *ex-interest bond*, the accrued interest is negative and in the example above (with settlement 20 August) the calculations are as follows:

Accrued interest price = – (d1 / 365 × c)
= – (26 / 365 × 12.0)
= – 0.85479.

Clean price = All-in price – accrued interest
= 94.60061 – (-0.85479)
= 94.60061 + R0.85479
= 95.45540.

5.5.4 Bonds with less than six months to maturity

In the case of bonds with less than six months to maturity, there are only two features to consider in pricing. They are:

- Number of days to maturity.
- The last coupon payment.

In this case the concept of clean price clearly does not arise because the price is termed the *all-in price*. The price of these short-term bonds is determined in terms of the following formula:

All-in price (PV)[32] = (1 + cr / 2) / [1 + (t / 365 × ir)]

where

cr = annual coupon rate
t = number of days from settlement date to maturity date
ir = interest rate (or ytm).

If the bond in the above example (a reminder: maturity date 15 September 2009; coupon 12%) were purchased on 21 July 2009 at 11.0% pa, its price would be as follows:

All-in price = (1 + 0.12 / 2) / (1 + (56 / 365 × 0.11))
= 1.04240757.

The consideration in this example would be as follows:

Consideration = nominal value × price
= LCC1 000 000 × 1.0424076
= LCC1 042 407.60.

However, if the bond has *less than one month to redemption* (i.e. is *ex interest*), the formula changes to the following:

All-in price (PV) = 1 / [1 + (t / 365 × ir)].

This is because the new purchaser does not receive the coupon and must be compensated for the loss of interest. Again using the above example, if the bond is purchased on 21 August 2009 at 11% pa, the all-in price would be as follows:

All-in price (PV) $= 1 / [1 + (25 / 365 \times 0.11)]$
$= 0.99252209.$

It will be apparent that the purchaser will receive LCC1 000 000 on 15 September (i.e. the nominal amount and no coupon payment). For this s/he pays a consideration of LCC992 522.09. S/he thus receives income of LCC1 000 000 less LCC992 522.09, or LCC7 477.90, for the 25-day period. His/her return (11.0% pa) can be recalculated as follows:

Return = (amount earned / price paid) × 365/25
= LCC7 477.90 / LCC992 522.09 × 365/25
= 0.1099999
= 11.0% pa.

5.6 Perpetual bonds

A perpetual bond is one that has no maturity date and therefore no repayment of the principal amount, and it pays a fixed annual (or more frequent) coupon rate. The price of such a bond is determined as follows:

$$\text{Price (PV)} = [cr / (1 + ytm)^1] + [cr / (1 + ytm)^2] + [cr / (1 + ytm)^3] + \ldots \textit{(infinity)}$$

Because infinity is involved here, this formula simplifies to the following:

$$\text{Price (PV)} = cr / ytm.$$

It should be clear that when cr = ytm, the price is 1.0 or 100%. For example, if the coupon rate is 10% and the ytm is 10%, the price is 10 / 10 = 1.0. If the market rate moves up to 20%, the price is 10 / 20 = 0.5 or 50%. If the rate moves down to 5%, the price is 10 / 5 = 2.0 or 200%.

The principle at work here is that when the market rate for the perpetual bond falls from 10% pa to 5% pa, the buyers are prepared to earn 5% pa in perpetuity. This means that they are *prepared to pay a price* for the security that will yield them 5% pa (= 200%) on a LCC1 million face value perpetual bond the annual income on which is LCC100 000 (10%). Thus, the buyers will be prepared to pay LCC2 000 000 for the bond (LCC100 000 / LCC2 000 000 × 100 = 5.0% pa).

It will have been noted that the inverse relationship between price and market rate is most easily understood in the case of a perpetual bond.

5.7 Bonds with a variable rate[33]

Bonds with a variable rate are also called floating rate bonds / floating rate notes (FRNs). They are issued for short- and long-term periods and pay a rate of interest (called a coupon) that is benchmarked on a well-publicised and available rate, for example IBAR[34] + 30 basis points (bp). This is an example called *simple margin* (or *spread for term*). Variations are adjusted simple margin, discount margin, etc.

It may be useful to start this discussion with an extreme example: that of a security that has a true-blue floating rate attached. The price of a security that has a coupon rate equal to, for example, 50 bp over the overnight IBAR rate (OIBAR), is always LCC100%. Thus if the OIBAR rate on day 1 is 7.0% pa, the rate on the bond = 7.5% pa, and its price is 1.0 or 100.00%. If the ROD rate falls to 6.5% pa on day 2, the price of the security remains at 100%. This is because there is no fixed rate that has a benefit for the holder. The rate fluctuates with what the market demands. Price risk is not present in this example.[35]

The above example is rare. Benchmark (also called *basis* and *reference*) rates used in respect of variable rate bonds are often ones that reset quarterly or six-monthly (for example 3-month IBAR yield + 30bp). It will be evident that the coupon rate is determined as follows:

Coupon rate = reference rate +/- quoted spread.

The spread is determined mainly by the borrower's credit rating (credit spread), term to maturity and the prevailing interest rate environment.

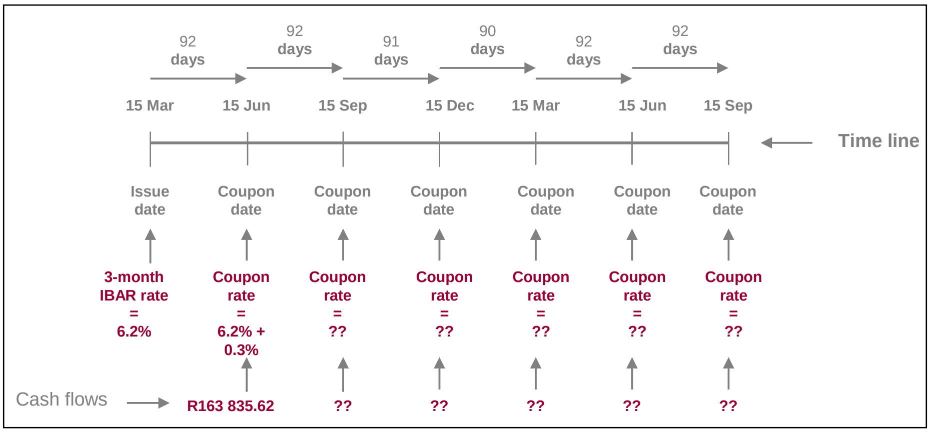

Figure 3: example of floating rate bond (LCC10 million, 3-month IBAR + 30bp)

In the case where coupons are payable quarterly, only the next coupon is certain; beyond this coupon payment the coupons are not known because the benchmark rates are not known (obviously) (see Figure 3).

These bonds [which are traded on price (all-in price) per LCC100 nominal] therefore do not – usually – trade at par (except at issue date and on rate reset dates).

The pricing (all-in price) of variable rate bonds is determined according to the following *inputs* (assumptions: quarterly reset dates; quarterly payments / payments are made on coupon dates):

- Initial spread (IS) – this is the margin at which the FRN is issued. For example, a FRN that bears interest at IBAR + 50bp has an IS of 50bp.
- Last interest reset rate (last rate on last coupon date).
- Number of days in the current interest period, i.e. number of days from the last coupon date to the next coupon date.
- Number of days from settlement date to next coupon date.
- Payment frequency of the bond (quarterly in the example).
- The values from a zero coupon yield curve[36] corresponding to the prospective coupon dates.
- Trading spread (the spread that would apply if the bond was issued on the day on valuation).

It will be evident that the pricing *methodology*, put simply, would be as follows:

- Generate a schedule of coupon payment dates (see example).
- Determine the zero coupon rate corresponding to each coupon date (from the zero coupon yield curve).
- Calculate the forward rate for each prospective coupon date (from the forward yield curve).
- Calculate the implied forward rates of the prospective coupons, i.e. predict the future coupons using the forward curve.
- Discount each coupon and principal back to present value, taking into account the market's perception of what the new trading spread should be.

In many cases, floating rate bonds have a cap, i.e. a maximum rate. If in this case rates are expected to move sideways or rise, the normal bond pricing formula applies.

5.8 CPI bonds

Bonds can also be issued at rates linked to an inflation rate. In this case, the floating (unknown) part of the interest can be paid to the holder or enhance the capital value of the bond.

An example is appropriate: a fixed coupon (for example, 5% pa) is paid six-monthly in arrears, and the capital value is enhanced by the rate of inflation. The following are the details:

Capital value (CV) of the bond $= P \times IR_{date}$

where

P = principal amount
IR_{date} = index ratio on relevant date.

The index ratio is calculated as follows:

$$IR_{date} = RCPI_{date} / RCPI_{idate}$$

where

$RCPI_{date}$ = reference CPI on date (usually interest-payment)
$RCPI_{idate}$ = reference CPI on issue date.

Coupons are thus paid on interest dates according to the formula:

Interest payment = (cr / 2) × CV

where

cr = coupon rate
CV = capital value.

In this example the coupon rate is 5% pa. Thus, on coupon dates the interest payable is 2.5% of the CV. If on an interest date the reference CPI is 119.9 and the reference CPI on issue date is 108.6, the index ratio is 1.10405 (119.9 / 108.6). Thus, the CV, assuming the principal amount is LCC10 000 000, is LCC11 040 500 (LCC10 000 000 × 1.10405). The coupon payable is thus LCC276 012.50 (0.025 × LCC11 040 500).

5.9 Zero coupon bonds

As noted above, zero coupon bonds are bonds that are discounted because they pay only the face value of the bond at the end of their life. The return earned by the holder is the difference between the amount paid, i.e. the discounted amount, and the face value of the bond.

A zero coupon bond is generally defined as follows[37]:

"A conventional bond is a debt instrument consisting of a series of periodic coupon payments plus the repayment of the principal on maturity. As the name suggests, a zero-coupon bond has no coupon payments. It has only a single payment consisting of the repayment of the principal at maturity. The zero-coupon bond is purchased by an investor at a discount to its face value and then redeemed for its face value at maturity. The return to the investor is the difference between the face value of the bond and its discounted purchase price."

The PV (price) of these bonds is calculated according to the normal bond pricing formula shown earlier. Thus, semi-annual compounding / discounting is assumed (in order to make them comparable to their plain vanilla counterparts). The valuation principle will be apparent from an example:

Face value = future value (FV)		= LCC1 000 000
Term to maturity in years	= y	= 3 years
Compounding periods pa	= cp	= 2
ytm		= 9% pa

$$
\begin{aligned}
PV &= FV / [1 + (ytm / cp)]^{y.cp} \\
&= LCC1\,000\,000 / [1 + (0.09 / 2)]^{3\times 2} \\
&= LCC1\,000\,000 / 1.045^{6} \\
&= LCC1\,000\,000 / 1.30226 \\
&= LCC767\,895.81.
\end{aligned}
$$

We may test this calculation by calculating the FV from the PV, using the same variables:

$$
\begin{aligned}
FV &= PV \times [1 + (0.09 / 2)]^{6} \\
&= LCC767\,895.81 \times 1.30226 \\
&= LCC1\,000\,000.00.
\end{aligned}
$$

It may be useful to present the valuation formulae in the case of (as these are used in some countries):

- Using number-of-days convention.
- Using annual compounding convention.

If the PV (price) of these bonds is calculated according to a day convention, the formula is:

$$PV = FV / (1 + ytm)^{d/365}$$

Example:

Face value (FV) = LCC1 000 000
D = number of days to maturity = 430 days
ytm = 9.35% pa

PV (or price) $= \text{LCC}1\,000\,000 / (1 + 0.0935)^{430/365}$
$= \text{LCC}1\,000\,000 / (1 + 0.0935)^{1.17808219}$
$= \text{LCC}1\,000\,000 / 1.11104519$
$= \text{LCC}900\,053.40.$

The buyer, using as givens the PV and the ytm, may verify this calculation as follows:

$$
\begin{aligned}
\text{FV} &= \text{PV}\,(1 + \text{ytm})^{d/365} \\
&= \text{PV}\,(1 + 1.0935)^{430/365} \\
&= \text{LCC}900\,053.40\,(1.0935)^{1.17808219} \\
&= \text{LCC}900\,053.40\,(1.11104519) \\
&= \text{LCC}1\,000\,000.00.
\end{aligned}
$$

In the case of unbroken years and *annual compounding* the calculation is:

Face value (FV) = LCC1 000 000
y = 3 years
ytm = 12.0% pa

$$
\begin{aligned}
\text{PV} &= \text{FV} / (1 + \text{ytm})^3 \\
&= \text{LCC}1\,000\,000 / (1.12)^3 \\
&= \text{LCC}1\,000\,000 / 1.4049280 \\
&= \text{LCC}711\,780.25.
\end{aligned}
$$

The test:

$$
\begin{aligned}
\text{FV} &= \text{PV}\,(1 + \text{ytm})^3 \\
&= \text{LCC}711\,780.25\,(1.12)^3 \\
&= \text{LCC}711\,780.25 \times 1.4049280 \\
&= \text{LCC}1\,000\,000.00.
\end{aligned}
$$

5.10 Strips

As noted in another section, STRIPS is the acronym for **S**eparate **T**rading of **R**egistered **I**nterest and **P**rincipal of **S**ecurities. A plain vanilla bond is stripped into C-strips and a P-strip. The amount and date of each interest payment (C-strips) and the final principal are known, and *each trades as a separate zero coupon bond* the mathematics of which was discussed above.

5.11 Summary

The basis underlying financial market mathematics is the time value of money, the PV / FV concept. The plain vanilla bond has the hallmarks of an annuity. A bond's value / price is determined by the discounting of all future cash flows to PV by the ytm. The latter is a "rate" created to cater for securities that have multiple future cash flows and can be described as an average rate for the period.

There are a number of variations to the plain vanilla bond, but the pricing principle remains the same.

5.12 Bibliography

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